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4) Prove that the angular spectrum method

$$U_{example}(x,y,z) = A(f_x, f_y; 0)\exp\left[i2\pi(f_x x + f_y y)\right]\exp\left[i2\pi z\sqrt{\left(\frac{k_0}{2\pi}\right)^2 - f_x^2 - f_y^2}\right]$$

Satisfies Helmholtz equation

Solution:

The scalar Helmholtz equation is

$$\nabla^2 U + k_0^2 U = 0$$

where $U(x, y, z)$ is the field and k_0 is the wavenumber $k_0 = \frac{2\pi}{\lambda}$.

Angular Spectrum Ansatz:

$$U(x,y,z) = A(f_x, f_y; 0)\exp\,[i2\pi(f_x x + f_y y)]\exp\left[i2\pi z\sqrt{\left(\frac{k_0}{2\pi}\right)^2 - f_x^2 - f_y^2}\right]$$

Substitute into the Helmholtz Equation:

Let

$$U(x,y,z) = A(f_x, f_y; 0)\exp\,[i2\pi(f_x x + f_y y)]\exp\,[i2\pi q z]$$

where

$$q = \sqrt{\left(\frac{k_0}{2\pi}\right)^2 - f_x^2 - f_y^2}$$

Calculate the Laplacian:

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

Each term:

- $\frac{\partial^2 U}{\partial x^2} = -(2\pi f_x)^2 U$
- $\frac{\partial^2 U}{\partial y^2} = -(2\pi f_y)^2 U$
- $\frac{\partial^2 U}{\partial z^2} = -(2\pi q)^2 U$

Total Laplacian:

$$\nabla^2 U = -(2\pi f_x)^2 U - (2\pi f_y)^2 U - (2\pi q)^2 U$$

Plug into Helmholtz equation:

$$[-(2\pi f_x)^2 - (2\pi f_y)^2 - (2\pi q)^2 + k_0^2]U = 0$$

Recall

$$q^2 = \left(\frac{k_0}{2\pi}\right)^2 - f_x^2 - f_y^2$$

$$(2\pi q)^2 = k_0^2 - (2\pi f_x)^2 - (2\pi f_y)^2$$

Thus:

$$-(2\pi f_x)^2 - (2\pi f_y)^2 - [k_0^2 - (2\pi f_x)^2 - (2\pi f_y)^2] + k_0^2 = 0$$

5) Test if the Huygens-Fresnel equation satisfies Helmholtz equation

$$U(P_0) = \frac{1}{i\lambda_0}\frac{\exp(ikr_{01})}{r_{01}}$$

$$r_{01} = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + z^2}$$

Solution:

The scalar Helmholtz equation is:

$$\nabla^2 U + k^2 U = 0$$

The Laplacian for a spherically symmetric function $U(r)$ is:

$$\nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right)$$

Compute first derivative:

$$\frac{\partial U}{\partial r} = \frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\right) = \frac{ike^{ikr}}{r} - \frac{e^{ikr}}{r^2}$$

Multiply by r^2:

$$r^2 \frac{\partial U}{\partial r} = ikre^{ikr} - e^{ikr}$$

Second derivative:

$$\frac{\partial}{\partial r}\left(ikre^{ikr} - e^{ikr}\right) = ike^{ikr} + ikr(ike^{ikr}) - ike^{ikr} = (ik)^2 re^{ikr} = -k^2 re^{ikr}$$

Divide by r^2

$$\nabla^2 U = \frac{-k^2 re^{ikr}}{r^2} = \frac{-k^2 e^{ikr}}{r}$$

Plug into Helmholtz Equation:

$$\nabla^2 U + k^2 U = (-k^2 \frac{e^{ikr}}{r}) + k^2 \frac{e^{ikr}}{r} = 0.$$

This is true everywhere except at $r = 0$, where the function is singular and the Laplacian yields a delta function representing a point source.

6) 6. Prove that the paraxial propagation transfer function is approximately equal to the angular spectrum transfer function for small angles.

$$H\left(f_x, f_y\right) = \exp\left[i2\pi z\sqrt{\left(\frac{k_0}{2\pi}\right)^2 - f_x^2 - f_y^2}\right] \approx \exp\left(ik_0 z\right)\exp\left[-i\pi\lambda_0 z\left(f_x^{\,2} + f_y^{\,2}\right)\right]$$

Solution:

Given the transfer function:

$$H(f_x, f_y) = \exp\left[i2\pi z\sqrt{\left(\frac{k_0}{2\pi}\right)^2 - f_x^2 - f_y^2}\right]$$

where $k_0 = \frac{2\pi}{\lambda_0}$.

For small f_x, f_y, we use the binomial (Taylor) expansion:

$$\sqrt{a^2 - x} \approx a - \frac{x}{2a} \text{ where}$$

Substitute a and x:

$$\sqrt{\left(\frac{k_0}{2\pi}\right)^2 - f_x^2 - f_y^2} \approx \frac{k_0}{2\pi} - \frac{f_x^2 + f_y^2}{2\frac{k_0}{2\pi}} = \frac{k_0}{2\pi} - \frac{\pi(f_x^2 + f_y^2)}{k_0}$$

Now substitute back into the exponential:

$$H(f_x, f_y) \approx \exp\left[i2\pi z\left(\frac{k_0}{2\pi} - \frac{\pi(f_x^2 + f_y^2)}{k_0}\right)\right]$$

$$= \exp\left[ik_0 z - i\pi\frac{2\pi z}{k_0}(f_x^2 + f_y^2)\right]$$

Recall $\lambda_0 = \frac{2\pi}{k_0}$:

$$H(f_x, f_y) = \exp(ik_0 z)\exp\left[-i\pi\lambda_0 z(f_x^2 + f_y^2)\right]$$

8. Prove that Maxwell's equations yield the wave equation.

Hint, take the curl of Faraday's law

Hint 2, use the identity $\nabla \times (\nabla \times \vec{\mathcal{E}}) = \nabla(\nabla \cdot \vec{\mathcal{E}}) - \nabla^2\vec{\mathcal{E}}$.

Solution:

Maxwell's Equations in Vacuum:

1. Gauss's law for electricity:

$$\nabla \cdot \mathrm{E} = 0$$

2. Gauss's law for magnetism:

$$\nabla \cdot \mathrm{B} = 0$$

3. Faraday's law of induction:

$$\nabla \times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t}$$

4. Ampere-Maxwell law:

$$\nabla \times \mathrm{B} = \mu_0 \varepsilon_0 \frac{\partial \mathrm{E}}{\partial t}$$

Using hint 1, take the curl of Faraday's law

$$\nabla \times (\nabla \times \mathrm{E}) = \nabla \times \left(-\frac{\partial \mathrm{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \times \mathrm{B})$$

Replace $\nabla \times \mathrm{B}$ with Ampere-Maxwell law, $\mu_0 \varepsilon_0 \frac{\partial \mathrm{E}}{\partial t}$:

$$\nabla \times (\nabla \times \mathrm{E}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathrm{E}}{\partial t^2}$$

Using hint 2 and applying the vector identity:

$$\nabla \times (\nabla \times \mathrm{E}) = \nabla(\nabla \cdot \mathrm{E}) - \nabla^2 \mathrm{E}$$

Since $\nabla \cdot \mathrm{E} = 0$ in vacuum (from Gauss's law), this simplifies to:

$$\nabla \times (\nabla \times \mathrm{E}) = -\nabla^2 \mathrm{E}$$

Substitute and rearrange:

$$-\nabla^2 \mathrm{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathrm{E}}{\partial t^2}$$

$$\nabla^2 \mathrm{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathrm{E}}{\partial t^2}$$

Final form of the vector wave equation for the electric field in free space:

$$\nabla^2 \mathrm{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathrm{E}}{\partial t^2} = 0$$

Additional notes:

- μ_0 is the vacuum permeability (magnetic constant).
- ε_0 is the vacuum permittivity (electric constant).
- The speed of light in vacuum is:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Therefore, the wave equation can also be written as:

$$\nabla^2 \mathrm{E} - \frac{1}{c^2} \frac{\partial^2 \mathrm{E}}{\partial t^2} = 0$$